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SATH SCRATCHES OF A MUPPET ON THE PLANE

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Abstract: The development of new methods and approaches to the representation of polynomial contour lines in the real plane using computer algebra can be of great theoretical and practical importance. It can contribute to the development of polynomial theory, analytic geometry, computer algebra and their interactions. In addition, the developed methods can be used in real applications such as scientific research, technical projects and solutions in various fields of science and technology.

Key words: power geometry, polynomial, polynomial contours, computer algebra.

Let $X = (x_1,x_2) \in R^2$. Consider the real polynomial $f(X)$. When $c \in R$ is constant, the curve in the plane R^2

$$f(X) = c \qquad (1)$$

is the level line of the polynomial f (X).

Let $C_* = \inf f(X)$ and $C^* = \sup f(X)$ by $X \in R^2$.

Theorem. There is a finite set of critical values c:

$$C_* < c_1 < c_2^* < \ldots < c_m^* < C^*, \qquad (2)$$

which correspond to the critical level lines

$$f(X) = c_j^*, j = 1, \ldots, m, \qquad (3)$$

and for the values of c from each of the m + 1 interval

$$I_o = (C_*, c_2^*), I_j = (c_j^*, c_{j+1}^*), j = 1,\ldots,m-1, I_m = (c_m^*, C^*) \qquad (4)$$

level lines are topologically equivalent. If $C^* = c_1^*$ or $C^* = c_m^*$, then there are no I_0 or I_m intervals.

Therefore to find the location of all level lines of the polynomial $f(X)$ it is necessary to find all critical values of c_j^*, depict m of critical level lines (3) and one level line for any value c of $m+1$ of the interval (4). The way to compute these level lines using power geometry is described in [1] and in part in [2, ch. I, § 2]. For a more traditional approach. The local structure of the polynomial level lines was considered in [2, ch. I, § 3]. Here some results from [2, ch. I, § 3] are supplemented.

A point $X = X^0$ is called *simple*, for a polynomial $f(X)$ if at least one of the partial derivatives $\partial f/\partial x_1$, $\partial f/dx_2$ is non-zero in it.

Definition 1. *Point* $X = X^0$ for a polynomial $f(X)$ is called *critical order k* if at the point $X = X^0$ all partial derivatives of $f(X)$ to order k are zero, that is, all

$$\frac{\partial^l f}{\partial x_1^i \partial x_2^j}(X^o) = 0, 1 \le i + j = l \le k,$$

and at least one partial derivative of order k + 1 is non-zero.

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Definition 2. *Curve*

$$g(X) = 0 \quad (5)$$

is called *critical* for a polynomial *f*(X) if

1. it lies on some level line (1) and
2. it has $\partial f/\partial x_1 = 0$, or $\partial f/\partial x_2 = 0$.

Let us call the values of the constant c = *f*(X) at the critical points $X = X^0$ and on the critical lines (5) as *critical* and denote c_j* according to (2).

Next, near the point $X = X^0$ we will consider analytic reversible coordinate substitutions

$$y_i = x_1^o + \varphi_i (x_1 - x_1^o, x_2 — x_2^o) ,i = 1,2, \quad (6)$$

where φ_i are- analytic functions from $X — X^0$.

Consider solutions to the equation (1) near the critical point $X^0 = 0$ of order 1. Then

$f(X) = f_0 + ax_1^2 + bx_1x_2 + cx_2^2 + ...$ (7)

The A discriminant of the quadratic form written out is $\Delta = b^2 — 4ac$.

Lemma l.[2, ch. I, § 3] If at the first-order critical point $X^0 = 0$ the discriminant of $\Delta \neq 0$, then there exists a substitution (6) that brings the equation (1) to the form

$f(X)=f_0 + \sigma y_1^2 + y_2^2 =c$, (8)

where σ=1 (if $\Delta < 0$) or $\sigma = -1$ (if $\Delta > 0$).

This is a two-dimensional version of Morse's famous lemma.

Lemma 2. If at the first-order critical point $X^0 = 0$ the discriminant $\Delta = 0$, then there exists a substitution (6) that brings the equation (1) to the form

$f(X)= f_0 + y_2^2 + \tau y_1^n = c$, (10)

where the integer $n > 2$ and the number $\tau \in \{—1, 0, +1\}$.

Let $X = (x_1, x_2) \in R^2$. Consider the real polynomial $f(X)$. At constant $c \in R$ the curve on the plane R^2

$f(X) = c$ (10)

is the line of the polynomial level $f(X)$.

Our task is to describe all the lines of the level of the polynomial $f(X)$ on the real plane $X \in R^2$.

Let $C_* = \inf f(X)$ and $C^* = \sup f(X)$ is $X \in R^2$.

The main result:

The theorem. There is a finite set of critical values *c:*

$C_* < c_1^* < c_2^* < \cdots < c_m^* < C^*$ (11)

which correspond to the critical lines of the level

$f(X) = c_j^*, j = 1, \dots, m$, (12)

a for the values of c from each of the $m + 1$ *interval*

$I_0 = (C_*, c_1^*), I_j = (c_j^*, c_j^* + 1), j = 1, \dots, m - 1, I_m = (c_m^*, C^*)$ (13)

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the level lines are topologically equivalent. If $C_* = c_1^*$ or $C^* = c_m^*$, then there are no intervals I_0 or I_m missing.

Therefore, to identify the location of all the lines of the level of the polynomial $f\,(X)$ it is necessary to find all the critical values of c_j^*, draw m critical lines of the level (2.1.3) and one level line for an arbitrary value of c from the $m + 1$ interval (2.1.4). he method of calculating these level lines is described in [1] and partly in [2, ch. I, § 2] using power geometry. For a more traditional approach, see [3, ch. I] or [4]. The local structure of the lines of the polynomial level was considered in [2, ch. I, § 3]. Here some results from [2, ch. I, § 3] are supplemented.

Critical points and critical curves

A point $X = X_0$ is called simple for a polynomial $f\,(X)$, if at least one of the partial derivatives $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}$. is nonzero in it.

Definition 1. A point $X = X_0$; for a polynomial $f\,(X)$ is called critical of order k, if at point $X = X_0$ all partial derivatives from $f\,(X)$ to order k, are equal to zero,

$$\frac{\partial^l f}{\partial x_1^i x_2^j}(X^0) = 0, 1 \leq i + j = l \leq k,$$

and at least one partial derivative of order $k + 1$ is nonzero.

Definition 2. Curve

$g(X) = 0$ (14)

is called critical for a polynomial $f\,(X)$, if

1. it lies on some level line (10) and
2. it $\partial f/\partial x_1 \equiv 0$, or $\partial f/\partial x_2 \equiv 0$.

The values of the constant $c = f\,(X)$ at the critical points $X = X_0$ and on the critical lines (14) are called critical and denoted c_j^* according to (11).

Local analysis of level lines

In the future, near the point $X = X^0$ we will consider analytical reversible substitutions of coordinates

$y_i = x_i^0 + \varphi_i(x_1 - x_1^0, x_2 - x_2^0)$, *i=1,2*, (15)

where φ_i — are analytic functions of $X = X^0$.

Lemma 1 ([2, ch. I, § 3])**.** *If the point X^0 is simple and in it $\partial f/\partial x_2 \neq 0$. then there is a replacement* (15), *which leads equation* (10) *to the form*

$f\,(X) = y_2 = c.$ (16)

It follows from the implicit function theorem.

The lines of level (16) — are straight lines parallel to the y_1.

Consider the solutions of equation (10) near the critical point

$X^0 = 0$ of order 1. Then

$$f(X) = f_0 + ax_1^2 + bx_1x_2 + cx_2^2 + \cdots$$

The discriminant Δ of the written quadratic form is $\Delta = b^2 - 4ac$.

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Lemma 2 ([2, гл. I, § 3])**.** If at the critical point of the first order

$X^0 = 0$ discriminant $\Delta = 0$, then there is a replacement (15), that leads equation (10) to the form

$f(X) = f_0 + \sigma y_1^2 + y_2^2 = c,$ (17)

where $\sigma = 1$ (*если* $\Delta < 0$) *or* $\sigma = -1$ (*if* $\Delta > 0$).

This is a two-dimensional version of the well-known Morse lemma [5, гл. I, § 2].

Lemma 3. If at the critical point of the first order $X^0 = 0$ the discriminant $\Delta = 0$, then there is a replacement (15) that leads equation (10) to the form $f(X) = f_0 + y_2^2 + \tau y_1^n = c,$ (16) *where the integer* $n > 2$ *and the number* $\tau \in \{-1, 0, +1\}$. Now let the critical point $X^0 = 0$ have the order $k > 1$. According to [1, раздел 5] the corresponding level line either has no branches entering the critical point $X^0 = 0$, or has several such branches.

In the first case, the critical level line consists of this point $X^0 = 0$, and the remaining level lines are closed curves around it and correspond to one sign of the difference $c - f(X^0)$.

In the second case, the critical level line consists of a finite number of branches of different multiplicities entering the critical point $X^0 = 0$. They divide the neighborhood of this critical point into curved sectors.

The remaining level lines fill these sectors, remaining at some distance from the critical point $X^0 = 0$. At the same time, in neighboring sectors, they correspond to different signs of the difference $c - f(X^0)$, if the branch separating them has an odd multiplicity, and to one sign of this difference if the branch separating them has an even multiplicity.

Definition 3. Let the edge $\boldsymbol{G}_j^{(1)}$ of the Newton polygon G(*f*) of the polynomial *g(X)=f(X) – c*, which has an external normal $\boldsymbol{P}_j^{(1)} = (\boldsymbol{p_1}, \boldsymbol{p_2})$ with one or two positive coordinates $\boldsymbol{p_i}$ and corresponds to the shortened polynomial $\boldsymbol{g}_j^{(1)}$ *(X).* Then the intersection of the root $\boldsymbol{x_1} = \boldsymbol{A\tau^{p_1}}, \boldsymbol{x_2} = \boldsymbol{B\tau^{p_2}}$ (where the constants *A*, *B* ∈ *R)* of the shortened polynomial $\boldsymbol{g}_j^{(1)}(\boldsymbol{X})$ for $\boldsymbol{\tau} \to \pm\infty$ c with infinity $\boldsymbol{x_i} = \pm\infty$ we call the infinite intersection point. The multiplicity of the root is the multiplicity of this point.

Lemma 4. If all edges $\boldsymbol{G}_j^{(1)}$ *from definition 3 do not contain points* $(\boldsymbol{q_1 q_2}) = \boldsymbol{0}$ *and the polynomial f*(*X*) *infinite intersection points have multiplicity one, then for all values of the constant c from one of the m* + 1 *intervals* (13) *of the line level (10) topologically equivalent.*

In general, it is convenient to calculate the critical values of c_j^* and their corresponding critical points and curves using computer algebra algorithms, primarily using the Groebner bases [6], [7,ch. 2, 3].

The ideal defining the critical points and curves consists of the following polynomials:

$$\mathcal{J} = \left\{ f(X) - c, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right\}. \quad (18)$$

As follows from the main theorem, the number of critical values of c_j^* is finite. Then the Groebner basis $\mathcal{GBJ}$ of the ideal (18) for the pure lexicographic order $c \succ x_1 \succ x_2$ contains a polynomial *h*(*c*), depending only on the variable *c*. Among the real roots of this polynomial, we should look for critical values of c_j^*, for which there are real critical level lines.

The Groebner basis, given by the $\mathcal{GBJ}$, ideal, has dimension either 0, or 1. In the first case, there are no critical curves, because the ideal $\mathcal{GBJ}$ is zero-dimensional. In the second case, some critical values of c_j^* correspond to critical curves.

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Most computer algebra systems have procedures for constructing Grobner bases for various lexicographic orders, as well as some additional procedures for checking the zero-dimensionality of an ideal, calculating its dimension, etc. Let's take a closer look at how such calculations can be implemented in the Maple system. You can use the Groebner and PolynomialIdeals packages, as well as the procedures included in it: Basis - to calculate the Groebner basis, Hilbert Dimension — to calculate the dimension of the ideal, IsZero Dimensional — to check the zero—dimensionality of the ideal, PrimaryDecomposition - for the primar decomposition of the ideal, Equidimension Decomposition — for the decomposition of the ideal into ideals of various dimensions.

We propose the following order of calculations.

1. The ideal *J* is computed and a Grobner basis *GBJ* with purely lexicographic order $x_1 < x_2 < c$ using the procedure Basis of Groebner package is computed for it.
2. The basis *GBJ* allows to find a1l critical values c_j^* using the polynomial h(c) as algebraic numbers. Such a polynomial is automatically determined for the above lexicographic order, or by the procedure UnivariatePolynomial.
3. Among the critical values of c_j^* one should select those to which real critical points and critical curves correspond. This can be done, for example, by performing a primal decomposition of the basis *GBJ* and then determining the zeros of the resulting ideals in the field R.
4. Before primary decomposition of an ideal *GBJ*, its dimensionality can be calculated using the procedure HilbertDimension. If it is not zero, first use the procedure EquidimensionalDecomposition to calculate a sequence of ideals with dimensions 0 and 1 respectively, and then perform their decomposition. Ideals of dimension 0 will yield critical points, and ideals of dimension 1 will yield critical curves (as well as real critical points).
5. Now for each primordial ideal find its set of real zeros. If the critical value c* belongs to Q, then all calculations are performed exactly. If it is an algebraic number, we can proceed as described in [1], i.e. all calculations are performed modulo the ideal defining the critical value and the critical point.
6. Then the nature of each critical point is determined using the discriminant A of the quadratic form of the expansion of the function *f*(X) near it according to Lemmas 2 and 3. The calculations performed allow us to proceed to the construction of sketches of the level lines.

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