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ON CONSTRUCTION OF REAL AW*-FACTORS

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Annotation: The paper of the is to initiate the study of real AW*-algebras in the framework of the theory of real C*-algebras and W*-algebras. It happens that in some aspects real AW*-algebras behave unlike complex AW*-algebras and sometimes their properties are completely different also from corresponding properties of real W*-algebras. We prove that if the complexification $A+iA$ of a real C*-algebra A is a (complex) AW*-algebra then A itself is a real AW∗-algebra. By modifying the Takenouchi's examples of complex non-W*, AW*-factors we show that there exist real non-W*, AW*-factors.

Keywords: AW*-algebra, C*-algebra, factor, involutive *-antiautomorphism, complex Hilbert space, commutant, complexification, linear *-automorphism, conjugate, bicommutant, quaternions algebra, projection, isomorphic.

1. INTRODUCTION

The theory of operator algebras was initiated in a series of papers by Murray and von Neumann in thirties. Later such algebras were called von Neumann algebras or W*-algebras. These algebras are self-adjoint unital subalgebras *M* of the algebra *B*(*H*) of bounded linear operators on a complex Hilbert space *H*, which is closed in the weak operator topology. Equivalently *M* is a von Neumann algebra in *B*(*H*) if it is equal to the commutant of its commutant (von Neumann's bicommutant theorem). A factor (or W*-factor) is a von Neumann algebra with trivial center and investigation of general W*-algebras can be reduced to the case of W*-factors, which are classified into types *I*, *II* and *III*.

Real operator algebra is a *-algebra consisting of bounded (real) linear operators on a real Hilbert space *H*. If it is closed in the weak operator topology we have real W*-algebra, and if it is uniformly closed (i.e. in the norm topology) then we come to the notion of the real C*-algebra.

In his monograph [7] Li Bing-Ren has set up the fundamentals of real operator algebras and gave a systematic discussion of the real counterpart for the theory of W*- and C*-algebras.

A slightly different (but almost the same up to *-isomorphism) definition of real W*-algebras was given by E.Størmer [13,14]: A real von Neumann algebra (or real W*-algebra) is a real *-algebra $\Re$ of bounded linear operators on a complex Hilbert space containing the identity operator **1**, which is closed in the weak operator topology and satisfies the condition $\Re \cap i\Re = \{0\}$.The smalles (complex) von Neumann algebra $U(\Re)$ containing $\Re$ coincides with its complexification $\Re \cap i\Re$, i.e. $U(\Re) = \Re \cap i\Re$. Moreover $\Re$ generates a natural involutive (i.e. of order 2) *-antiautomorphism $\alpha_{\Re}$ of $U(\Re)$, namely $\alpha_{\Re} = (x+iy) = x^* + iy^*$, where $x+iy \in U(\Re)$, $x, y \in \Re$. It is clear that $\Re = \{x \in U(\Re) : \alpha(x) = x^*\}$. Conversely, given a (complex) von Neumann algebra *U* and any involutive *-antiautomorphism *α* on *U*, the set $\{x \in U : \alpha(x) = x^*\}$ is a real von Neumann algebra in the above sense.

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It is not diffucul to see that two real von Neumann algebras generating the same (complex) von Neumann algebra are isomorphic if and only if the corresponding involutive *-antiautomorphisms are conjugate. Thus the study of the above real von Neumann algebra can be reduced to the study of pairs (*U, α*), where *U* is a (complex) von Neumann algebra and *α* - its involutive *-antiautomorphism.

2. PRELIMINARIES

Let *H* be a complex Hilbert space, $B(H)$ denote the algebra of all bounded linear operators on *H*. The *weak (operator) topology* on $B(H)$ is the locally convex topology, generated by semi norms of the form: $\rho(a)=|(\xi,a\eta)|,\ \xi,\eta\in H,\ a\in B(H)$. W*-algebra is a weakly closed complex *-algebra of operators on a Hilbert space *H* containing the identity operator **1**. Recall that W*-algebras are also called *von Neumann algebras*.

Let further *M* be a W*-algebra. The set M' of all elements from $B(H)$ commuting with each element from *M* is called the commutant of the algebra *M*. The *center* $Z(M)$ of a W*-algebra *M* is the set of elements of *M*, commuting with each element from *M*. It is easy to see that $Z(M)=M\cap M'$. Elements of $Z(M)$ are called *central* elements. A W*-algebra *M* is called *factor*, if $Z(M)$ consists of the complex multiples of **1**, i.e if $Z(M)=\{\lambda 1,\ \lambda\in\mathbb{C}\}$. We say that a W*-algebra *M* is *injective* if thereexists a projection *P* in $B(H)$ onto *M* such that $\|P\|=1$ and $P(1)=\mathbf{1}$. This isequivalent to the condition that *M* is *hyperfinite*, i.e., that there exists an increasing sequence $\{M_n\}$ of matrix subalgebras of the algebra *M* containing **1** and such that the union $\cup_n M_n$ is weakly dense in *M*.

Let e, f, h be projections from *M*. We say that e is equivalent to f, and write $e\sim f$, if $e=\omega^*\omega,\ f=\omega\omega^*$ for some partial isometry ω from *M*. A projection e is called: *finite*, if $e\sim f\leq e$ implies $f=e$; *infinite* - otherwise; *purely infinite*, if e doesn't have any nonzero finite subprojection; abelian, if the algebra eMe is an abelian W*-algebra. A W*-algebra *M* is called *finite, infinite, purely infinite*, if **1** is a finite, infinite, purely infinite respectively; *M* is σ-finite, if any family of pairwise orthogonal projections from *M* is at most countable; *semifinite*, if each projection in *M* contains a nonzero finite subprojection; *properly infinite*, if every nonzero projection from $Z(M)$ is infinite; *discrete*, or of type *I*, if it contains a faithful abelian projection (i.e. an abelian projection with the central support **1**); *continuous*, if there is no abelian projection in *M* except zero; *M* is of type *II*, if *M* is semifinite and continuous; type I_{fin} (respectively I_∞), if *M* is of type *I* and finite (respectively properly infinite); type II_1 (respectively type II_∞), if *M* is of type *II* and finite (respectively properly infinite); type *III*, if *M* is purely infinite. It is known that (see for example [103]) any W*-algebra has a unique decomposition along its center into the direct sum of W*-algebras of the $I_{fin}, I_\infty, II_1, II_\infty$ and *III* types.

A linear mapping $\alpha: M\to M$ is called a **-automorphism* (respectively a *-antiautomorphism) if $\alpha(x^*)=\alpha(x)^*$ and $\alpha(xy)=\alpha(x)\alpha(y)$ (respectively $\alpha(xy)=\alpha(y)\alpha(x)$, for all $x,y\in M$. A mapping α is called involutive if $\alpha^2=id$. A *-automorphism α is called *inner* if there exists a unitary u in *M*, such that $Adu(x)=uxu^*$, for all $x\in M$. A *-automorphism is called centrally trivial if $\alpha(x_n)-x_n\to 0$ *-strongly as $n\to\infty$ for any central sequence $\{x_n\}_{n\in\mathbb{N}}$. We shall denote by

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$Aut(M)$ the group of all *-automorphisms, by $Aut(M)$ the group of all *antiautomorphisms, by $Int(M)$ the group of all inner *-automorphisms, and by Ct(*R*) the subgroup of its centrally trivial *-automorphisms of *M*. Two *-automorphisms or *-antiautomorphisms α and β are said to be conjugate (or outer conjugate), if $\alpha = \theta \cdot \beta \cdot \theta^{-1}$ (respectively $Adu \cdot \alpha = \theta \cdot \beta \cdot \theta^{-1}$) for some *-automorphism θ (and an inner *-automorphism *Adu*). A linear functional ω on *M* is called *positive*, if $\omega(x^*x) \geq 0$ for all $x \in M$. A positive linear functional ω on *M* with $\|\omega\| = 1$ is called a state. Let M_+ be the positive part of *M*. A *weight* on *M* is a homogeneous additive function $\omega : M_+ \to [0, +\infty]$ (we suppose that $0 \cdot +\infty = 0$). A weight (or a state) ω is called: *faithful*, if for any $x \in M_+$, $\omega(x) = 0$ implies $x = 0$; *normal*, if for any net $\{x_\alpha\}$ in *M*, increasing to an element *x*, we have $\omega(x) = \sup_\alpha \omega(x_\alpha)$; *finite*, if $\omega(x) < \infty$ for all $x \in M_+$; *semifinite*, if for any $x \in M_+$ there exists a net of elements $\{y_\alpha\} \in M_+$, such that $\omega(y_\alpha) < \infty$, and $y_\alpha \to x$ in σ - weak topology; ω is a trace, if $\omega(uxu^*) = \omega(x)$ for all $x \in M_+$ and each unitary $u \in M$.

The type of a W*-algebra is tightly connect with the existence of traces on it. Namely a W*-algebra *M* is a finite if and only if it possesses a separating family of finite normal traces; it is semifinite if and only if it possesses a faithful semifinite normal trace; *M* is purely infinite if and only if there is no nonzero semifinite normal trace on M (see [15]).

Definition [4]. By a real C*-algebra we mean a real Banach *-algebra *R* such that the relation $\|a^*a\| = \|a\|^2$ holds and the element $1 + a^*a$ is invertible for any $a \in R$.

Definition* [8,9]. A real C*-algebra *R* such that *R*+*iR* is a complex W*-algebra is referred to as a real W*-algebra.

We proceed with another definition of a real W*-algebra, which can be found in papers of Størmer.

Definition** [2,13]. A unital weakly closed real *-algebra *R* in $B(H)$ such that $R \cap iR = \{0\}$ is called a *real W*-algebra*.

A real W*-algebra *R* is called a (real) factor if its center *Z*(*R*) consists of elements $\lambda\mathbf{1}$, $\lambda \in$ □ **.** We say that a real W*-algebra *R* is of type I_{fin}, I_∞, II_1, II_∞ and *III*, $\lambda \in [0,1]$ if the enveloping W*-algebra $U(R) = R + iR$ (i.e., the least W*-algebra containing *R*) is of the corresponding type with respect to the usual classification of W*-algebras.

3. MAIN RESULTS

Let A be a real C*-algebra, with the complexification $M = A + iA$. Then *M* is a complex C*-algebra and, as we have seen in the previous section, if A is a real AW*-algebra *M* may not be a (complex) AW*-algebra. Now let us consider the converse problem if $M = A + iA$ is an AW*-algebra is A necessarily a real AW∗-algebra? The following result gives a positive answer to this problem.

Proposition 1. *Let A be a real C*-algebra and let* $M = A + iA$ *be its complexification. Suppose that M is an AW*-algebra. Then A is a real AW*-algebra.*

Proof. As we have mentioned in the first section, A coincides with the fixed point set under the conjugate linear *-automorphism $"-": x + iy \mapsto x - iy$ of *M*, where $x, y \in A$, i.e.

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$A=\{a\in M:\bar{a}=a\}$. If S is a nonempty subset in A then for its right-annihilator (with respect to M) we have

$$R_M(S)=\{a\in M\,|\,sa=0 \ \ for\ all\ s\in S\}$$

and

$$a\in R_M(S)\Leftrightarrow sa=0,\forall s\in S\Leftrightarrow \overline{sa}=\bar{s}\,\bar{a}=s\bar{a}=0,\ \forall s\in S$$

because $\bar{s}=s\in A$. This means that $a\in R_M(S)$ if and only if $\bar{a}\in R_M(S)$.

Now suppose that M is an AW*-algebra, then $R_M(S)=gM$ for a suitable projection $g\in M$. Since $g\in R_M(S)$ from above it follows that $\bar{g}\in R_M(S)$. Therefore $\bar{g}$ is a projection and $\bar{g}\in gM$, i.e. $\bar{g}=g\bar{g}$. Thus $(\bar{g})^*=(g\bar{g})^*=(\bar{g})^*g^*=\bar{g}g=\bar{g}=g\bar{g}$ i.e. $g=\bar{\bar{g}}=\overline{g\bar{g}}=\bar{g}g=\bar{g}$

This means that $g\in A$. But then

$$R_M(S)=R_M(S)\cap A=gM\cap A=gA,$$

i.e. A is a real AW*-algebra.

Proposition 2. *There exist real AW*-factors which are not real W*-factors.*

Theorem 1. *A real AW*-algebra A is a real W*-algebra if and only if*

1. *A possesses a separating family of normal states;*
2. *its complexification* $M=A+iA$ *is an AW*-algebra.*

Proof. Necessity is obvious, since if A is a real W*-algebra, then $M=A+iA$ is a (complex) W*-algebra (see [7, Chap.5]). Therefore, M is an AW*-algebra and it possesses a separating family of normal states, the restrictions of which on A give a separating family of normal states on A.

Sufficiency. Let $M=A+iA$ be an AW*-algebra and let A possess a separating family of normal states, which we denote by $\{f_\gamma\}$, i.e. for any $a\in A,\ a\geq 0,\ a\neq 0$ exists $f\in\{f_\gamma\}$ with $f(a)=0.$

For $x\in a+ib\in M,\ a,b\in A,$ we put $\alpha(x)=a^*+ib^*.$ A straightforward calculation shows that α is an involutive (i.e. with period 2) *-anti-automorphism of M, and $A=\{a\in M:\alpha(a)=a^*\}$.

The extension of f_γ by linearity on M we denote by f_γ^0, and we shall show, that the family $\{f_\gamma^0\}$ is a separating family of normal states on M.

For $x=a+ib\in M_s=\{x\in M:x^*=x\}$ we have $a^*=a,\ b^*=-b$, and since f_γ is hermicitian we obtain $f_\gamma^0(x)=f_\gamma(a)+if_\gamma(b)=f_\gamma(a),$ since $f_\gamma(b)=0$. Thus, for $x\in M_s$ we have

$$f_\gamma^0(x)=\frac{1}{2}f_\gamma(x+\alpha(x)),$$

and $x+\alpha(x)\in A,$ since $x+\alpha(x)=2a.$

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If $x \geq 0$, then $\alpha(x) \geq 0$ (because α is a *-anti-automorphism), and hence $x + \alpha(x) \geq 0$, i.e. $x + \alpha(x) \in A^{+}$. Therefore $f_\gamma^0(x) = \frac{1}{2} f_\gamma(x + \alpha(x)) \geq 0$, i.e. all functionals f_γ^0 are positive on *M*. Moreover, we have $f_\gamma^0(1) = f_\gamma(1) = 1$, i.e. $\{f_\gamma^0\}$ is a family of states on *M*.

Now let us show, that each state f_γ^0 is normal. If $\{x_\nu\} \subset M$ is an arbitrary net with $x_\nu \square\ 0$, then since α is an order isomorphism of *M*, we have $\alpha(x_\nu) \square\ 0$. Therefore $x_\nu + \alpha(x_\nu) \square\ 0$ and $x_\nu + \alpha(x_\nu) \in A^{+}$. Since f_γ is a normal we obtain

$$f_\gamma^0(x_\nu) = \frac{1}{2} f_\gamma(x_\nu + \alpha(x_\nu)) \to 0$$

i.e. all functionals f_γ^0 are normal on *M*.

Finally, let $x \in M$, $x \geq 0$ and $f_\gamma^0(x) = 0$ for all γ. Then $x + \alpha(x) \in A^{+}$, and since $\{f_\gamma\}$ is a separating family of states, $x + \alpha(x) = 0$. Hence we have $x = -\alpha(x) \in M^{+} \cap (-M^{+}) = \{0\}$, i.e. $x = 0$. Thus, the AW*-algebra *M* possesses a separating family of normal states $\{f_\gamma^0\}$. By the theorem of Pedersen [10] *M* is a W∗-algebra. Therefore, by [7] *A* is a real W∗-algebra.

Now, let *M* be a (complex) AW*-factor, α its involutive *-anti-automorphism. Then as it was mentioned above the set $A = \{a \in M : \alpha(a) = a^*\}$ is a real C*-algebra such that $M = A + iA$ (actually $\overline{x} = \alpha(x^*)$ in terms of operation ”-”) and from Proposition 1 it follows that *A* is a real AW*-factor. It is known that two real W*-algebras generating the same (complex) W*-algebra, are isomorphic if and only if the corresponding involutive *-anti-automorphisms are conjugate [2,13,14]. A similar result is also valid for real AW*-algebras:

Theorem 2. *Let α and β be involutive *-anti-automorphisms of a (complex) AW*-factor M. Then the real AW*-factors*

$$A = \{x \in M : \alpha(x) = x^*\} \text{ and } B = \{x \in M : \beta(x) = x^*\}$$

*are real *-isomorphic if and only if the involutive *-anti-automorphisms α and β are conjugate, i.e.* $\beta = \theta\alpha\theta^{-1}$ *for a suitable *-automorphism of the AW*-factor M.*

Proof. Let *A* and *B* be real *-isomorphic with a *-isomorphism $\theta_0 : A \mapsto B$. Then θ_0 can be naturally extended to a (complex) *-isomorphism θ of their complexifications $A + iA$ and $B + iB$ both coincide with *M*. Therefore θ is a *-automorphism of *M* and $\theta(A) = B$, i.e. $\alpha(x) = x^*$ if and only if $\beta(\theta(x)) = (\theta(x))^* = \theta x^*$. Thus for $x \in A$ we have

$$\beta(\theta(x)) = (\theta(x))^* = \theta(x^*) = \theta\alpha(x) \text{, i.e. } \beta = \theta\alpha\theta^{-1}\alpha(x) \text{ for all } x \in A.$$

Since $\theta^{-1}\beta^{-1}\theta\alpha$ is a *-automorphism on *M* which is identical on *A* and any real *automorphism of *A* can be uniquely extended to a complex *-automorphism of *M*, it follows that $\theta^{-1}\beta^{-1}\theta\alpha = id$ on whole *M*, i.e. $\theta\alpha = \beta\theta$ and $\beta = \theta\alpha\theta^{-1}$, i.e. α and β are conjugate.

Conversely, if α and β are conjugate, i.e. $\beta = \theta\alpha\theta^{-1}$ for a suitable complex *-automorphism θ of *M*, then $\theta\alpha = \beta\theta$ and $\alpha(x) = x^*$ if and only if $\beta(\theta(x)) = \theta x^* = (\theta x)^*$, i.e. $\theta(A) = B$. Therefore, θ restricted on *A* gives the needed *-isomorphism between real AW*-factors *A* and *B*.

REFERENCES

1. Ayupov Sh.A., Azamov N.A. Commutators and Lie isomorphisms of skew elements in prime operator algebras. Comm. Algebra, 1996, 24, N 4, pp.1501-1520.
2. Ayupov Sh.A., Rakhimov A.A., Usmanov Sh.M. Jordan, Real and Lie Structures in Operator Algebras. MAIA, Kluwer Academic Publishers. 1997. Vol.418, 235p.
3. Berberian S.K. Baer *-rings. Springer-Verlag, BerlinHeidelbergN.Y. 1972.
4. Isidro J.M., Palacios A., Rodriguez. Proc. Amer. Math. Soc., 1996, N11, Vol.124, pp. 3407-3410.
5. Kaplansky I. Projections in Banach algebras. Ann. Math., 1951, 53, pp.235-249.
6. Kaplansky I. Algebras of type I., Ann. Math. 1952, 56, pp. 460-472.
7. Li Bing-Ren. Real operator algebras. World Scientific Publishing Co. Pte. Ltd. 2003. 241p.
8. Li Bing-Ren. Introduction to Operator Algebras. World Sci.. Singapore., 1992. pp. 237-256.
9. Li Minli, Li Li Bing-Ren. Acta Math. Sinica (N.S.), 1998. N1, Vol.14, pp. 85-90.
10. Pedersen G. Operator algebras with weakly closed abelian subalgebras. Bull. London Math. Soc. 1972, 4, pp. 171-175.
11. PutnamC.R.OnnormaloperatorsinHilbertspaces.Amer.J.Math.1951,Vol.73, pp. 357-362.
12. Sakai S. Operator Algebras in Dynamical Systems. Encyclopedia of Mathematics and its Applications, 1991, Vol. 41, Cambridge University Press, Cambridge.
13. Størmer E. On anti-automorphisms of von Neumann algebras. Pacific J. Math., 1967, N2, Vol.21, pp 349-370.
14. Størmer E. Real structure in the hyperfinite factor. Duke Math. J., 1980. Vol. 47. N 1. pp. 145-153.
15. Takesaki M. Theory of Operator Algebras. I, Springer-Verlag. 1979. VIII + 415p.