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The Spectrum of the Sum of Two Orthogonal Projectors in Separable Gilbert Space

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Abstract. This paper considers the spectrum of two orthogonal projectors in separable Gilbert space. In separable Gilbert space, the appearance of the sum of the spectrum of two orthogonal projectors is calculated. It is also shown that the self-adjoint operator A can be expressed as a linear combination of two orthoprojectors.

Key words: Gilbert space, separable Gilbert space, orthogonal projectors, linear combination, orthoprojectors, *-homomorphisms, subspaces, self-adjoint operator, orthogonal spaces.

1. INTRODUCTION

The theory of unitary representations of groups dates back to the 19th century and is associated with the names of G. Frobenius, I. Schur, W. Burnside, F.E. Molina and others. In connection with proposals to quantum physics, the theory of unitary representations of topological groups, Lie groups, C*-algebras was developed by I.M. Gelfand, M.A. Naimark, I. Segal, J. Dixmier, A.A. Kirillov and others in the 70-80s of the XX century [see. 1-3]. Later, the theory of representations of *-algebras given by generators and relations was intensively developed.

Let $\mathbf{H}$ be a Hilbert space and $\mathbf{L}(\mathbf{H})$ be a set of continuous linear operators in $\mathbf{H}$. Consider a $\mathbf{A}$ subset in $\mathbf{L}(\mathbf{H})$ that is preserved under addition, multiplication, multiplication by scalars, and conjugation. Then $\mathbf{A}$ is an operator *-algebra. Given an abstract *-algebra $\mathbf{A}$, then one of the main problems in the theory of linear representations (*-homomorphisms $\mathbf{A}$ in $\mathbf{L}(\mathbf{H})$) is to enumerate all its irreducible representations (up to equivalence) [see. 4-6].

2. FORMULATION OF THE PROBLEM

Theorem 1. (spectrum theorem). There is only one invariant expansion of H with respect to P_1 *and* P_2 in the form of an orthogonal sum of subspaces

$$\mathbf{H} = \mathbf{H}_{0,0} \oplus \mathbf{H}_{0,1} \oplus \mathbf{H}_{1,0} \oplus \mathbf{H}_{1,1} \ ((\mathbb{C}^2 \oplus \mathbf{H}_\kappa)),$$

here, let there be one $\varphi_\kappa \in (0, \frac{\pi}{2}),\ \varphi_\kappa \neq \varphi_i\ (k \neq j),\ \dim \mathbf{H}_\kappa = n_\kappa\ \ (\kappa = 1, \dots, m).$

$P_{i,j} : \mathbf{H} \rightarrow \mathbf{H}_{i,j}$ operator and $P\varphi_\kappa : \mathbf{H} \rightarrow \mathbb{C}^2 \oplus \mathbf{H}_\kappa$ $(\kappa = 1, \ldots, m)$ orthoprojector for each $\mathbf{H}_\kappa$ subspace. Then there is a unique spread of operators

$$\mathbf{I} = \mathbf{P}_{0,0} \oplus \mathbf{P}_{0,1} \oplus \mathbf{P}_{1,0} \oplus \mathbf{P}_{1,1} \oplus (\oplus_{k=1}^{m} \mathbf{P}\varphi_\kappa),$$

$$P_1 = \mathbf{P}_{1,0} \oplus \mathbf{P}_{1,1} \oplus (\oplus_{k=1}^{m} (\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus \mathbf{I}_k)),$$

$$P_2 = \mathbf{P}_{0,1} \oplus \mathbf{P}_{1,1} \oplus (\oplus_{k=1}^{m} (\begin{pmatrix} \cos^2 \varphi_k & \cos\varphi_k \sin\varphi_k \\ \cos\varphi_k \sin\varphi_k & \sin^2 \varphi_k \end{pmatrix} \oplus \mathbf{I}_k)),$$

where $\mathbf{I}_k$ – $\mathbf{H}_\kappa$ $(\kappa = 1, \ldots, m)$ is the unary operator.

Let **H** be two orthoprojectors in separable Gilbert space. Then we know the spectrum of each of them:

$$P_1 = \mathbf{P}_{1,0} \oplus \mathbf{P}_{1,1} \oplus (\oplus_{k=1}^{\infty} (\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbf{I}_k)) \text{ and}$$

$$P_2 = \mathbf{P}_{0,1} \oplus \mathbf{P}_{1,1} \oplus (\oplus_{k=1}^{\infty} (\begin{pmatrix} \cos^2 \varphi_k & \cos\varphi_k \sin\varphi_k \\ \cos\varphi_k \sin\varphi_k & \sin^2 \varphi_k \end{pmatrix} \otimes \mathbf{I}_k)).$$

We can irreducible the spectrum of the sum $P_1 + P_2$ in the following representations.

3. MAIN RESULTS

Theorem 2. A self-adjoint operator *A* can be expressed as the sum of two orthoprojectors $A = P_1 + P_2$ if $\sigma(A) = [0,\ 2]$ and the space **H** is partitioned into an orthogonal sum of invariant spaces *A*

$$\mathbf{H} = \mathbf{H_0} \oplus \mathbf{H_1} \oplus \mathbf{H_2} \oplus (\oplus_{k=1}^{\infty} (\mathbb{C}^2 \otimes \mathbf{L}_2((0, \frac{\pi}{2}), d\rho_\kappa))) \qquad (1)$$

and is invariant under a change of ρ_κ measures $1 + x \rightarrow 1 - x$.

Proof. Let $A = P_1 + P_2$ and $\mathbf{H_0} = \mathbf{H_{0,0}}$, $\mathbf{H_1} = \mathbf{H_{1,0}} \oplus \mathbf{H_{0,1}}$, $\mathbf{H_2} = \mathbf{H_{1,1}}$. $\varphi \rightarrow \varepsilon cos\varphi$ we introduce the following match, where $\varphi \in (0, \frac{\pi}{2})$. Then, as found above, the spectrum $\sigma(A) \in [0,\ 2]$ and **H** (based on the spectral theorem of $[4]$) can be decomposed into an orthogonal sum:

$$\mathbf{H} = \mathbf{H_0} \oplus \mathbf{H_1} \oplus \mathbf{H_2} \oplus (\oplus_{k=1}^{\infty} (\mathbb{C}^2 \otimes \mathbf{L}_2((0,2), d\rho_\kappa))).$$

Since the eigenspaces corresponding to the eigenvalues of *A* - $1+\varepsilon,\ 1-\varepsilon$ $(0 < \varepsilon < 1)$ are simultaneously included in the spectrum and their values coincide, each $\rho_\kappa(\kappa = 1,\ 2,\ \ldots)$ measure $1 + x \rightarrow 1 - x$ is invariant through transformation.

Contrariwise. (1) and let $\sigma(A) \in [0,\ 2]$ be a proper function. Then we determine the orthoprojectors by P_1', P_2' equations

$$P_1' = \mathbf{P}_1 \oplus \mathbf{P}_2 \oplus (\oplus_{k=1}^{\infty}(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbf{I}_k)), \qquad (2)$$

$$P_2' = \mathbf{P}_1 \oplus \mathbf{P}_2 \oplus (\oplus_{k=1}^{\infty}(\begin{pmatrix} \cos^2 \varphi_k & \cos\varphi_k \sin\varphi_k \\ \cos\varphi_k \sin\varphi_k & \sin^2 \varphi_k \end{pmatrix} \otimes \mathbf{I}_k)), \qquad (3)$$

where $\mathbf{P}_i : \mathbf{H} \to \mathbf{H}_i \ \ (i = \ 0,\ 1,\ 2)$ is the orthoprojector, and $\mathbf{I_k}$ - $\mathbf{L_2}((0,2), d\rho_\kappa))$ is the unit operator. Then $A \ = P_1' + \ P_2'$ is a self-adjoint operator whose spectrum lies in $[0,\ 2]$, since $P_\kappa' \ \ (\kappa = \ 1,\ 2)$ is the sum of orthoprojectors into mutually orthogonal spaces.

Now let's look at the state of $A = aP_1 + bP_2 \quad (0 < a < b)$.

Theorem 3. A self-adjoint operator *A* can be expressed as a linear combination of two orthoprojectors $A = aP_1 + bP_2 \quad (0 < a < b)$, if $\sigma(A) \subset \ [0,\ a] \cup [b,\ a+b]$ and **H** can be expressed as an orthogonal sum of the space *A*.

$$\mathbf{H} = \mathbf{H}_{(\mathbf{0})} \oplus \mathbf{H}_{(a)} \oplus \mathbf{H}_{(b)} \oplus \mathbf{H}_{(a+b)} \oplus (\otimes_{k=1}^{\infty}(\Box^{2} \otimes \mathbf{L_2}([0,a] \cup [b,a+b], d\rho_\kappa))), \quad (4)$$

and is invariant under a change of ρ_κ measures $x \to a+b$.

Proof. Let $A = aP_1 + bP_2 \quad (0 < a < b)$ and $\mathbf{H_0} = \mathbf{H_{0,0}}$, $\mathbf{H_1} = \mathbf{H_{1,0}} \oplus \mathbf{H_{0,1}}$, $\mathbf{H_2} = \mathbf{H_{1,1}}$. $\sigma(A) \subset \ [0,\ a] \cup [b,\ a+b]$ and the eigenspaces corresponding to the eigenvalues of the operator *A* simultaneously enter into **H** (and they have the same dimensions), then, as in Theorem 1:

$$\mathbf{H} = \mathbf{H}_{(\mathbf{0})} \oplus \mathbf{H}_{(a)} \oplus \mathbf{H}_{(b)} \oplus \mathbf{H}_{(a+b)} \oplus (\otimes_{k=1}^{\infty}(\Box^{2} \otimes \mathbf{L_2}([0,a] \cup [b,a+b], d\rho_\kappa))),$$

where $\rho_\kappa\,(\kappa \ = \ 1,\ 2,\ \ldots)$ measures are invariant under $x \to a+b-x$ transformation.

Contrariwise, $\sigma(A) \subset \ [0,\ a] \cup [b,\ a+b]$ and let **H** have the spread (4). Then we set P_1 and P_2 as follows:

$$\mathbf{P}_1 = \mathbf{P}_a \oplus \mathbf{P}_{a+b} \oplus (\oplus_{k=1}^{\infty}(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbf{I}_k)),$$

$$\mathbf{P}_2 = \mathbf{P}_b \oplus \mathbf{P}_{a+b} \oplus (\oplus_{k=1}^{\infty}(\begin{pmatrix} \cos^2 \varphi_k & \cos\varphi_k \sin\varphi_k \\ \cos\varphi_k \sin\varphi_k & \sin^2 \varphi_k \end{pmatrix} \otimes \mathbf{I}_k)),$$

where $\mathbf{P}_\alpha : \mathbf{H} \to \mathbf{H}_\alpha \ \ (\alpha = a,\ \mathrm{b},\ a+b)$ is the orthoprojector, and $\mathbf{I_k}$ -

$\mathbf{L}_2([0,a]\cup[b,a+b])$ is the unit operator. In that case

$$a\mathbf{P_1}+b\mathbf{P_2}=a\mathbf{P_a}\oplus b\mathbf{P_b}\oplus(a+b)\mathbf{P_{a+b}}\oplus(a\oplus_{k=1}^{\infty}(\begin{pmatrix}1&0\\0&0\end{pmatrix}\otimes\mathbf{I}_k))\oplus$$

$$\oplus(b\begin{pmatrix}\cos^2\varphi_k & \cos\varphi_k\sin\varphi_k\\ \cos\varphi_k\sin\varphi_k & \sin^2\varphi_k\end{pmatrix}\otimes\mathbf{I_к}))=\mathbf{A}.$$

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Conclusion

Using the example of the sum of the spectrum of orthoprojectors arranged in the case of unitary spaces, the sum of the spectrum of orthoprojectors is also found in separable Gilbert spaces.

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